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Thermal Waves in Absorbing Media

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We discuss the existence of travelling-wave solutions with interfaces for the nonlinear heat equation with absorption

$$u_t = a(u^m)_{xx} - bu^n$$

with a, $b > 0$ and m, $n \in \mathbb{R}$. Several situations occur depending on the relative strength of the diffusion and absorption terms reflected by their exponents m and n. We characterize the existence of finite travelling waves in terms of m and n, show their uniqueness up to translations in space and time, and derive their velocity from the wave profile near the interface or front. © 1988 Academic Press, Inc.

1. INTRODUCTION

We study here the nonlinear parabolic equation

$$u_t = a(u^m)_{xx} - bu^n, \qquad a, b > 0, \tag{1.1}$$

for different values of parameters m and n. Equation (1.1) is a simple and widely used model for various physical problems involving diffusion and absorption, as for instance transport of thermal energy in plasma (cf. [RK]).

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We are interested in a particular class of solutions to (1.1), the *finite travelling waves* (FTW). By a travelling-wave solution (TW) with velocity $c \in \mathbb{R}$ we mean a solution $u(x, t)$ of (1.1) in $Q = \{(x, t): -\infty < x < +\infty, t > 0\}$ of the form:

$$u(x, t) = \phi(ct - x), \tag{1.2}$$

where $\phi(\xi) \geqslant 0$, $\phi \not\equiv 0$, and $\phi(\xi) \to 0$ as $\xi \to -\infty$. In case $\phi(\xi) = 0$ for $\xi \leqslant \xi_0$ and some $\xi_0 \in \mathbb{R}$ we say that u is a finite travelling wave. A FTW with positive (resp. negative) velocity c is called in thermal propagation a *heating wave*, HW (resp. a *cooling wave*, CW). The case $c = 0$ obviously corresponds to stationary solutions, SW.

Equation (1.1) has a number of interesting features not present in the classical heat equation $u_t = u_{xx}$ nor, more generally, in uniformly parabolic equations. Perhaps the most relevant, and the one we shall focus our attention on, is the property of *finite propagation.* This phenomenon has been extensively studied in the absence of absorption, i.e., for $b = 0$, when $m > 1$. Setting $a = 1$ we thus obtain the so-called porous medium equation:

$$u_t = (u^m)_{xx}. \tag{1.3}$$

Let us consider, to be specific, the Cauchy problem for (1.3) with initial data $u(x, 0) = u_0(x)$ nonnegative, vanishing outside of a bounded interval $(\alpha, \beta) \subset \mathbb{R}$ and such that $u_0^{m-1} \in C^1[\alpha, \beta]$ and $(u_0^{m-1})'(x)$ does not vanish at $x = \alpha$ and $x = \beta$. Then, this problem has a unique weak solution $u(x, t)$ in $Q = \mathbb{R} \times (0, \infty)$, u is continuous, and for every $t > 0$, u is positive on an interval $(S_1(t), S_2(t))$, where the functions $S_i \in C^1([0, \infty))$ and $(-1)^i S_i$ are strictly increasing in $\mathbb{R}_+$, $i = 1, 2$. The curves $x = S_i(t)$, usually called interfaces or free boundaries, represent in transport problems the propagation fronts, and their properties are of much interest and have been extensively studied (cf., e.g., [P, AV] and their references).

Two properties of the interfaces are worth mentioning. First, not only the support of the solution $[S_1(t), S_2(t)]$ is strictly *expanding* in time, but also $|S_i(t)| \to \infty$ as $t \to \infty$; i.e., the whole space is reached by the propagation phenomenon.

The second property consists in relating the velocity of the interfaces, $S_i'(t)$, to the shape of the solution near them. In fact, the following equation holds [A, Kn, CF],

$$S_i'(t) = -\lim \frac{m}{m-1} \frac{\partial}{\partial x} (u^{m-1}), \tag{1.4}$$

the limit being taken as $x \to S_i(t)$ with $u(x, t) > 0$. In the framework of gas flow through porous media, the function in the second member represents the local velocity of diffusion of the gas [A], and (1.4) expresses the fact that diffusion is the only cause for the movement of the fronts.

The finite propagation property of (1.3) with $m>1$ is also reflected in the existence of FTWs as we shall see below. On the contrary, the property fails if $m \leqslant 1$. It is in fact wellknown that for $m=1$ there is infinite speed of propagation, reflected in the fact that a nonnegative solution of $u_t = u_{xx}$ in a subdomain Ω of Q is in fact positive everywhere. An analogous result holds true when $m<1$ [AB], even if $m \leqslant 0$, in which case we write $u_t = (u^{m-1} u_x)_x$ instead of (1.3) [H, ERV].

The introduction of an absorption term of the form $-F(u)$ may have a deep influence on the qualitative behavior of the solutions. In the case where $F(u) = bu^n$ with $b>0$ and $n \geqslant m > 1$, the picture is basically unchanged. In particular, finite propagation holds, fronts expand unboundedly, [Ka1, Ke1, Kn] and (1.4) controls the movement of the interfaces [HV]. For $1 \leqslant n < m$ solutions with compact support in x penetrate only to a finite distance in the medium as $t \to \infty$ (localization), but the support still expands with time.

Dramatic changes occur for $n<1$, in which case the volumetric absorption rate bu^{n-1} blows up as $u \to 0$. For the first time we encounter phenomena like extinction in finite time [Ka2, FH], pulse splitting [RK], or occurrence of cooling fronts [Ke2, M], the latter being of interest here. In fact, there exist solutions with shrinking interfaces and even solutions whose fronts change direction as time passes, first expanding and then receding.

In [RK], the behavior of the solutions to (1.1) near a moving free boundary is formally analyzed for $m>1$ and $0<n<1$. Assuming that an interface $x = S(t)$ separates a hot region $u>0$ to its left from a cold region $u=0$ to the right, an expansion

$$u(x,t) = K[(x - S(t_0) - c(t-t_0)]^l + \cdots \tag{1.5}$$

is tried for the solution of (1.7) in the region $x < S(t)$ near a point $(S(t_0), t_0)$ at the front. Three possible profiles arise from this asymptotical analysis, depending on the values of m and n, and the equations that relate $S'(t)$ to any such profile are also formally derived.

In this paper, we perform a rigorous investigation on the existence and properties of finite travelling waves for all possible values of the exponents m and n and the velocity c. While it is easy to see that stationary solutions of this type exist whenever $m>0$ and $n<|m|$, cooling waves occur if $m>0$ and $-m<n<\min(m,1)$, and heating waves exist for $m>0$ and $n<|m|$, like the stationary waves, and also for $n=m>1$. When $n>m>1$ a class of heating waves which blow up for $\xi \uparrow \xi_\infty < \infty$ is also obtained. There is no limitation as to the velocity c with which these waves move, and for any fixed c the corresponding wave is unique up to translations in space and time: this is the sense in which uniqueness will be understood henceforth. As for propagation , we show that the three profiles predicted in [RK] actually occur for suitable ranges of the parameters, which we determine. The equation for the velocity of the front is derived in each case. On the other hand, the waves obtained are unbounded functions, and we analyze their growth rate as $\xi \to \infty$. We also prove that the profiles depend continuously and monotonically on the velocity c.

We end this Introduction with a few remarks. First, Eq. (1.1) is invariant not only under space and time translations, but also under transformation $(x, t) \mapsto (-x, t)$. Applying this symmetry to the above TW's we get another family of TW's that now vanish as $x \to -\infty$ and grow with x. This completes all of the possible nonnegative solutions of the form (1.2). Second, though we have performed our analysis in $1+1$ dimensions, the results can be viewed as the study of planar travelling waves in $N+1$ dimensions (i.e., solutions of the form

$$u(x, t) = \phi(ct - v \cdot x),$$

with $t > 0$, $c \in \mathbb{R}$, $x \in \mathbb{R}^N$, and v a unit vector in $\mathbb{R}^N$) for the equation

$$u_t = a\Delta(u^m) - bu^n.$$

2. Different Profiles: The Case $m + n = 2$

Let us derive in simple particular situations the three different wave profiles that will appear in our investigation. The first such case corresponds to the model of diffusion in the absence of absorption, i.e., when $a > 0$ and $b = 0$. One obtains for $m > 1$ the family of heating waves

$$v(x, t) = \frac{c}{a}(ct - x)_+, \qquad c > 0, \tag{2.1}$$

where $v = m/(m-1)\, u^{m-1}$, the equation at the interface being then given by (1.4). A FTW such that $(u^{m-1})_x$ admits a negative limit as $\zeta = ct - x \to 0$ (resp. as $\xi \to \infty$), $\xi > 0$, is said to have a *diffusive profile* or *D-profile* at the front (resp. at infinity). It is easy to check that no FTW exists when $m \leqslant 1$.

The second profile appears when we consider the opposite situation where the diffusion term is disregarded, i.e., for $a = 0$, $b > 0$. One easily obtains for $-\infty < n < 1$ the family of cooling waves

$$w(x, t) = -\frac{b}{c}(ct - x)_+, \qquad c < 0, \tag{2.2}$$

where $w = u^{1-n}/(1-n)$. The equation at the interface is now given by

$$c = b \lim_{ct - x \downarrow 0} \left(\frac{1}{w_x}\right). \tag{2.3}$$

A FTW has an *absorptive profile* or *A-profile* at the front if the limit on the right hand side of (2.3) exists and is negative. The A-profile at infinity is defined in a similar way.

The third type or S-profile corresponds to the stationary solutions. These appear just by dropping the term u_t in (1.1) and looking for equilibrium configurations in the form of a FTW. Equation $a(u^m)_{xx} = bu^n$ admits a nonnegative solution in $\mathbb{R}$ vanishing for $x > 0$ but not for $x < 0$ if and only if $m > 0$ and $|n| < m$. In that case it is given by

$$\gamma u(x, t)^{(m-n)/2} = \left(\frac{b}{a}\right)^{1/2} (-x)_+, \qquad \gamma = \frac{(2m(m+n))^{1/2}}{m-n}. \tag{2.4}$$

Notice that all these functions are C^∞ in the region where $u > 0$, but fail to be smooth in general at the front $x = ct$. Therefore the concept of the solution has to be understood in the sense of distributions (cf. [OKC, HV]). This is equivalent in our case to saying that u is a classical solution of (1.1) in the region $ct - x > 0$ such that u and $(u^m)_x$ vanish on the front $x = ct$.

It is worth noting that the above exponents $m-1$, $1-n$, and $(m-n)/2$ coincide when $m + n = 2$. If we write (1.1) in terms of v in this critical case we get, for $m > 1$,

$$v_t = (m-1)\, avv_{xx} + av_x^2 - bm, \tag{2.5}$$

for which a finite travelling solution exists for every $c \in \mathbb{R}$. It has the form

$$v(x, t) = \lambda(ct - x)_+ \qquad \lambda > 0, \tag{2.6}$$

where λ(i.e., $-v_x$ in the region $u > 0$) is related to the velocity c by the equation

$$c = \lambda a - \frac{bm}{\lambda}. \tag{2.7}$$

We therefore get a heating wave for $\lambda > \lambda_0 = (bm/a)^{1/2}$, a cooling wave for $\lambda < \lambda_0$, and the stationary solution (2.4) for $\lambda = \lambda_0$.

3. The ODE Setting: Heating Waves for $m+n>2$

In showing the existence of heating waves of the form (1.2) for Eq. (1.1), we are faced with the following problem: to find a continuous, real function $\phi\colon [0, \infty) \to \mathbb{R}$ such that

$$\phi(\xi)>0 \qquad \text{for} \quad \xi>0 \quad \text{and} \quad \phi(0)=0, \tag{3.1.a}$$

$$a(\phi^m)''=c\phi'+b\phi^n \qquad \text{in the sense of distributions in } (0, \infty), \tag{3.1.b}$$

$$(\phi^m)'(0)=0. \tag{3.1.c}$$

We shall say that two positive functions $\phi(\xi)$, $\psi(\xi)$ defined in $(0, \infty)$ are equivalent as $\xi \to 0$ (resp. as $\xi \to \infty$), and we shall write $\phi \cong \psi$, if $\phi(\xi)/\psi(\xi) \to 1$ as $\xi \to 0$ (resp. as $\xi \to \infty$). One then has:

Theorem 1. *Let $2-m<n\leqslant m$. Then for any $c>0$ there exists a unique heating wave $u_c(x, t)$ of the form* (1.2) *such that $\phi^{m-1} \in C^\infty(0, \infty) \cap C^1[0, \infty)$. Moreover,*

$$\frac{m}{m-1}(\phi^{m-1})' \cong \frac{c}{a} \qquad \textit{as} \quad \xi \to 0, \tag{3.2}$$

whereas, as $\xi \to \infty$,

$$\gamma(\phi^{(m-n)/2})' \cong \left(\frac{b}{a}\right)^{1/2}, \qquad \textit{with } \gamma \textit{ given in } (2.4), \textit{ if } n<m, \tag{3.3.a}$$

$$m(\log \phi)' \cong \left(\frac{b}{a}\right)^{1/2} \qquad \textit{if} \qquad n=m. \tag{3.3.b}$$

Proof. We shall show that a unique $\phi(\xi)$ satisfying (3.1) exists such that (3.2) and (3.3) hold. To this end, we shall use a phase-plane argument. Let us introduce the variables:

$$X=\phi^m, \qquad Y=(\phi^m)'. \tag{3.4}$$

Then our problem can be reformulated as finding the nontrivial trajectories of the differential system

$$\begin{aligned} X' &= Y \\ aY' &= \frac{c}{m}\, YX^{-(m-1)/m} + bX^{n/m} \end{aligned} \tag{3.5}$$

which start from $(0, 0)$ at $\xi=0$, exist for $0<\xi<\infty$, and are contained in the first quadrant $\Omega_1=\{(X, Y)\colon X>0, Y>0\}$ for $\xi>0$. We claim that there exists one and only one such trajectory, $\bar{Y}(x)$. To show this, we first note that the equation for the trajectories is

$$\frac{dY}{dX}=\frac{c}{am}X^{-(m-1)/m}+\frac{b}{a}X^{n/m}\cdot Y^{-1}\equiv F(X,Y). \tag{3.6}$$

To begin with, uniqueness follows from the lucky fact that $\partial F/\partial Y=-(b/a)X^{n/m}$. $Y^{-2}<0$ whenever $Y\neq 0$. This implies that if two trajectories, say $Y(x)$ and $\tilde{Y}(x)$, pass respectively through points $(X_1, \tilde{Y}_1)$ and (X_1, Y_1) with $Y_1>\tilde{Y}_1$, then for $0<X<X_1$ we have $Y(x)>\tilde{Y}(x)$ and the difference $(Y(x)-\tilde{Y}(x))$ is strictly increasing as $X\downarrow 0$. Therefore, Y and $\tilde{Y}$ cannot meet for $X<X_1$. This argument will hold for all the wave types to be considered in the sequel.

As to the existence, this comes from a topological argument. Let us shoot from points on the boundary of Ω_1. It is clear then that there exists a unique solution of (3.6) starting from each point $(A,0)$ with $A>0$. To see that there also exists a unique solution in Ω_1 starting from a point $(0, B)$ with $B>0$ it suffices to remark that

$$\frac{dX}{dY}\cong\frac{am}{c}X^{(m-1)/m}$$

for $(X, Y)\in\Omega_1$ near $(0, B)$, and the exponent $(m-1)/m$ is less than one. Since $F(X, Y)\geqslant 0$ in Ω_1, all these solutions are monotone curves and they form a nonintersecting ordered family. We find then $\bar{Y}(x)$ as the supremum of the family of solutions $Y_A(x)$ passing through $(A,0)$ with $A>0$, or equivalently as the infimum of the $Y_B(x)$ starting from $(0, B)$ with $B>0$.

We now estimate the behavior of $\bar{Y}(x)$ near $(0,0)$. Heuristic arguments suggest that the important terms in (3.6) are then Y' and $(c/am)X^{1/m-1}$, corresponding to u_t and $a(u^m)_{xx}$, thus producing a diffusive profile while the last term is of higher order. To prove this, we use the comparison function $Z(X)=MX^{1/m}$ with $M>0$. If $M>c/a$, we have

$$Z'(X)=\frac{M}{m}X^{-(m-1)/m}>\frac{c}{am}X^{-(m-1)/m}+\frac{b}{M}X^{(n-1)/m}\equiv F(X, Z(X))$$

for X small enough. Since $Z(0)=\bar{Y}(0)$, we deduce that $\bar{Y}(X)\leqslant MX^{1/m}$ if $M>c/a$ and $X\cong 0$. In a similar way, we can show that $\bar{Y}(X)\geqslant MX^{1/m}$ if $M<c/a$ and $X>0$. Summing up, we get

$$\bar{Y}(X)\cong\frac{c}{a}X^{1/m} \qquad \text{for} \quad X\cong 0. \tag{3.7}$$

Let us prove next that $\bar{Y}$ corresponds to a finite wave. Suppose the trajectory passes through $(X_1 Y_1) \in \Omega_1$ for $\xi = \xi_1$. Then, if we call ξ_0 the value of ξ for which $(X(\xi), \bar{Y}(\xi))$ starts from $(0, 0)$, we have by (3.4)

$$\xi_1 - \xi_0 = \int_0^{X_1} \frac{dX}{Y} \cong \frac{a}{c} \int_0^{X_1} s^{-1/m}\, ds = \frac{am}{c(m-1)} X_1^{(m-1)/m} < \infty. \tag{3.8}$$

Therefore $\xi_0 > -\infty$, and after a translation we may fix $\xi_0 = 0$. Observe that the argument depends on estimate (3.7) plus the fact that $m > 1$. Now (3.8) is just another way of saying that $(m/(m-1))\, \phi^{m-1} \cong (c/a)\, \xi$ for $\xi \to 0$. As for the derivative estimate, notice that by (3.4),

$$\left(\frac{m}{m-1} \phi^{m-1}\right)' = YX^{-1/m} \cong \frac{c}{a} \qquad \text{as} \quad \xi \to 0 \tag{3.9}$$

so that (3.2) holds.

Let us now study the behavior as $X \to \infty$. Under our current assumption $n + m > 2$, the relevant terms in (3.6) will be Y' and $(b/a)\, X^{n/m} \cdot Y^{-1}$, corresponding to an equilibrium profile. To wit, we try $Z(X) = MX^{\alpha}$ with $M > 0$ and $\alpha = (n+m)/2m$ as a comparison function. It then follows that $Z'(X) > F(X, Z(X))$ for large X if $M^2 > (b/a\alpha)$ and $Z'(X) < F(X, Z(X))$ for all large X if $M^2 < (b/a\alpha)$ with F as before, since the term $(c, am)\, X^{1/m-1}$ is now of lower order. Therefore we get

$$\bar{Y}(X) \cong \left(\frac{2mb}{a(n+m)}\right)^{1/2} X^{(n+m)/2m} \qquad \text{as} \quad X \to \infty. \tag{3.10}$$

From (3.10) estimate (3.3) follows by integrating $X'(\xi) = Y(\xi)$ as in (3.8) and noting that

$$\left(\frac{2m}{m-n} \phi^{(m-n)/2}\right)' = YX^{-(n+m)/2m}. \quad \blacksquare \tag{3.11}$$

The restriction $n \leqslant m$ in the above proof comes only in the last argument, to show that $\xi \to \infty$ as X and Y go to $+\infty$, so that we have a wave in all of Q. When $n > m$, the exponent in (3.10) satisfies $(n+m)/2m > 1$, so that

$$\xi_\infty = \int d\xi = \int_0^\infty \frac{dX}{Y(X)} < +\infty. \tag{3.12}$$

This means that for $n > m > 1$ we can construct a finite travelling wave in the region $\{(x, t) \in Q : x - ct > -\xi_\infty\}$ with a front at $x = ct$ but blowing up as $x - ct \downarrow -\xi_\infty$. A further computation with $X' = Y$ shows that this blowup is of the form

$$\phi(\xi) \cong C(\xi_\infty - \xi)^{-2/(n-m)} \qquad \text{as} \quad \xi \uparrow \xi_\infty, \tag{3.13}$$

where $C = C(a, b, m, n)$. On the other hand, we recall that condition $m > 1$ is necessary to ensure that the wave is finite (see (3.8)). We have thus obtained the following result.

THEOREM 2. *If $n > m > 1$ there is a unique partial heating wave in the region $\{(x, t): x - ct > -\xi_\infty\}$ where ξ_∞ is given by* (3.12). *Moreover, the wave profile satisfies* (3.2) *as $\xi \to 0$ and blows up as $\xi \uparrow \xi_\infty$ as described in* (3.13).

Remark. The comparison argument in Theorem 1 can be used to obtain second order terms in the description of the wave profile. Thus we can get that

$$\bar{Y}(X) = \frac{c}{a} X^{1/m} + \frac{bm}{c(m+n-1)} X^{(m+n-1)/m} + \cdots$$

as $X \to 0$, i.e.,

$$\left(\frac{m}{m-1} \phi^{m-1}\right)' = \frac{c}{a} + \frac{bm}{c(m+n-1)} \phi^{m+n-2} + \cdots$$

as $\xi \to 0$, from which it follows that

$$\frac{m}{m-1} \phi^{m-1} = \frac{c}{a} \xi(1 + K\xi^{(m+n-2)/(m-1)} + \cdots),$$

where $K = K(a, b, c, m, n) > 0$.

4. COOLING WAVES FOR $m + n > 2$

We now look for solutions to (1.1) in the form

$$u_c(x, t) = \begin{cases} \phi(ct - x) & \text{if } x < ct, \\ 0 & \text{if } x \geqslant ct, \end{cases} \tag{4.1}$$

with a negative velocity c. This amounts to finding $\phi(\xi)$ satisfying (3.1.a)–(3.1.c), the only difference with the previous section consisting in the sign of c.

We have

THEOREM 3. *Let $2 - m < n < 1$. Then for any $c < 0$ there exists a unique cooling wave $u_c(x, t)$ of the form* (4.1) *such that $\phi^{1-n} \in C^\infty(0, \infty) \cap C^1[0, \infty)$. Moreover*

$$\frac{1}{1-n} (\phi^{1-n})' \cong -\frac{b}{c} \qquad as \qquad \xi \to 0 \tag{4.3}$$

and

$$\gamma(\phi^{(m-n)/2})' \cong \left(\frac{b}{a}\right)^{1/2} \qquad \textit{as} \quad \xi \to \infty. \tag{4.4}$$

Proof. As in the proof of Theorem 1, we introduce the variables X, Y and arrive at system (3.5), whose trajectories are the solutions of (3.6).

Uniqueness is shown as in Theorem 1. As for existence, we remark that now $F(X, Y) > 0$ in the subdomain D of the first quadrant Ω_1 limited by the axis $Y=0$ and the curve

$$H(X) = \frac{bm}{|c|} X^{(m+n-1)/m}, \tag{4.5}$$

while $F<0$ in $\Omega_1 - \bar{D}$. Since $dY/dX \to \infty$ as $(X, Y) \to (A, 0)$, $A>0$, it is again a consequence of standard topological arguments that there exists a nontrivial trajectory $\tilde{Y}(X)$ contained in D, i.e., such that

$$0 < \tilde{Y}(X) < H(X)$$

and approaching $(0, 0)$ as ξ decreases. To study the behavior of $Y(X)$ near the origin we again use a comparison function, whose choice is motivated by a heuristic argument suggesting that in our case the effect of diffusion is negligible, and the first-order approximation is given by the last two terms in (3.6). Namely, we try $Z(X) = MX^{(n+m-1)/m}$ with $M>0$. Then it is easy to see that near $X=0$ we have $Z'(X) > F(X, Z(X))$ (resp. $Z'(X) < F(Z, Z(X))$) if $M > -bm/c$ (resp. $M < -bm/c$). We thus conclude that

$$Y(X) \cong -\frac{bm}{c} X^{(n+m-1)/m} \qquad \text{as} \quad X \to 0. \tag{4.6}$$

Again, to prove that the trajectory hits $(0, 0)$ for finite ξ we check that

$$\xi = \int_0^X \frac{dX}{Y(X)} \cong \frac{|c|}{bm} \int_0^X s^{-(n+m-1)/m}\, ds < +\infty, \tag{4.7}$$

which happens precisely for $n<1$. We obtain from (4.7) an expression for $\phi = X^{1/m}$ in terms of ξ, namely,

$$\frac{\phi(\xi)^{1-n}}{1-n} \cong -\frac{b}{c}\xi \qquad \text{as} \quad \xi \cong 0.$$

As to (4.3), it follows from (4.6) and the identity

$$\frac{1}{1-n}(\phi^{1-n})' = \frac{1}{m} YX^{(n+m-1)/m}. \quad \blacksquare$$

Finally, the behavior as $\xi \to \infty$ is obtained exactly as in Theorem 1.

Remark. Changing ξ in $-\xi$, and consequently (X, Y) into $(X, -Y)$ allows us to transform the study of Eq. (3.6) with parameter c in Ω_1 into the study of (3.6) with parameter $-c$ in the fourth quadrant: $X>0$, $Y<0$. In this way, the half-plane, $X>0$ in (3.6) gives both a heating and a cooling wave.

5. The Case $m+n<2$

In this Section we prove that heating and cooling waves exist when $m+n<2$ if and only if moreover $|n|<m$. This last condition ensures the existence of stationary solutions whose profile near the front coincides in the first approximation, as we show, with that of both heating and cooling waves. The influence of the different velocities will only appear as a perturbation of the S-profile. With the notation of previous Sections, one has:

THEOREM 4. *Let $m+n<2$. Then travelling waves exist, with exactly one for each velocity $c \in \mathbb{R}$, if and only if $|n|<m$. In this case, the wave profile is C^∞ where positive, and*

$$\gamma(\phi^{(m-n)/2})' \cong \frac{b}{a} \qquad \textit{as} \quad \xi \to 0 \tag{5.1}$$

with γ given in (2.4). *Moreover*

$$\frac{m}{m-1}(\phi^{m-1})' \cong \frac{c}{a} \qquad \textit{as} \quad \xi \to \infty \textit{ for heating waves} \tag{5.2}$$

$$\frac{1}{1-n}(\phi^{1-n})' \cong -\frac{b}{c} \qquad \textit{as} \quad \xi \to \infty \textit{ for cooling waves.} \tag{5.3}$$

Proof. Let us begin with the statement corresponding to heating waves with velocity $c>0$. The argument in this case is entirely similar to that in Theorem 1, the main point again being that

$$\frac{dX}{dY} = \frac{amYX}{cYX^{1/m} + bX^{(n+m)/m}} \cong \frac{amBX}{cBX^{1/m} + bX^{(n+m)/m}} \tag{5.4}$$

near points $(0, B)$ with $B>0$ provided that $m+n>0$. A comparison with test functions of the form $Z(X) = MX^\alpha$ with $\alpha = (n+m)/2m$ shows then that we have an S-profile near the origin, i.e.,

$$\tilde{Y}(X) \cong \left(\frac{2bm}{a(m+n)}\right)^{1/2} X^{(n+m)/2m} \qquad \text{as} \quad X \to 0, \tag{5.5}$$

whence the front is finite and ϕ satisfies (5.1) precisely if $n<m$.

In the case of cooling waves, the existence argument parallels that in Theorem 3 if $n+m>1$, since in that case $H(X)$ given in (4.5) is still a curve starting at (0, 0). However, if $m+n<1$, $H(X)$ decreases from ∞ to zero as X increases from the origin. Then the existence of the nontrivial trajectory reaching (0, 0) depends, as in the case of heating waves, on the existence in the domain D (now bounded not only by the curve $Y=H(X)$ and the axis $Y=0$, but also by the axis $X=0$) of trajectories emanating from points of the form $(0, B)$ with $B>0$. Again, it comes from (5.4) that they exist when $m+n>0$. When $m+n=1$, we have $H(X)=-bm/c$, and existence follows as in the latter argument, this time applied for $0<B<-bm/c$.

On the other hand, the behavior as $X\to\infty$ comes from comparison with test functions of the form $Z(X)=MX^{\alpha}$ with $\alpha=1/m$ for heating waves and $\alpha=(m+n-1)/m$ for cooling waves. This yields (5.2), (5.3). Finally, uniqueness follows just as in Section 3. ∎

It is interesting to remark that, in describing the wave profiles, the roles of $\xi\to\infty$ and $\xi\to 0$ are just reversed when passing from $m+n>2$ to $m+n<2$. In particular, the asymptotic behavior as $\xi\to\infty$ of both heating and cooling waves for $m+n>2$ is given in first approximation by the stationary solution, and so is the profile near the front for $m+n<2$. This means that the velocity c of the particular wave is not reflected in the leading order term in the last case. We next show how c is related to the second-order approximation of the profile near the front.

THEOREM 5. *Let $m+n<2$ and $-m<n<m$. If ϕ is the profile of a travelling wave with velocity $c\in\mathbb{R}$ we have as $\xi\to 0$*

$$\gamma(\phi^{(m-n)/2})(\xi)\cong\left(\frac{b}{a}\right)^{1/2}\xi(1+ck\xi^{\delta}), \tag{5.6}$$

where $\delta=(2-m-n)/m-n$ and $k=k(a, b, m, n)$.

Proof. It suffices to show the theorem in the case of heating waves. As in the remark after Theorem 2, we use a two term test fuction $Z(X)$ to get a better approximation to $\bar{Y}(X)$ near the origin. We try

$$Z(X)=MX^{\alpha}+NX^{\beta} \tag{5.7}$$

and use for M and α the correct values for an S-profile as obtained in (5.5). Plugging (5.7) into (3.6) and arguing as above we finally arrive at

$$\bar{Y}(X)\cong\left(\frac{b}{a}\right)^{1/2}\frac{m}{\gamma}X^{(m+n)/2m}+\frac{2c}{a(3m+n)}X^{1/m}.$$

Using (3.11), (5.6) follows in our case. Cooling waves are dealt with in a similar way. ∎

COROLLARY. *On the above assumptions on m and n, the equation at the interface $x=S(t)$ for the FTW is*

$$\frac{dS}{dt}=\lim \frac{(u_x^\alpha)(x,t)-\sigma}{\mu\,|ct-x|^\delta}, \tag{5.8}$$

where $\alpha=(m-n)/2$, $\delta=(2-m-n)/(m-n)$, $\sigma=(b/a)^{1/2}\gamma^{-1}$, $\mu=\mu(a,b,m,n)$, *and the limit is taken as* $(ct-x)\to 0$ *with* $x<ct$.

EXAMPLE. Assume $n+m\equiv 1$. In this case, the ODE (3.6) is explicitly integrable. In particular, the nontrivial solution passing through $(0,0)$ is given by the formula

$$\bar{Y}(X)=\frac{bm}{c}\left\{\exp\left(\frac{c}{bm}\left(\bar{Y}(X)-\frac{c}{a}X\right)^{1/m}\right)-1\right\}. \tag{5.9}$$

Expanding the second member of (5.9) up to second-order terms, we can easily obtain (5.5) as $x\to 0$.

The above analysis shows that ϕ always behaves as $\xi\to 0$ or $\xi\to\infty$ like a power of ξ, the three possible exponents being ordered as corresponds to the physical situation: $1/(m-1)<2(m-n)<1/(1-n)$ if $m+n>2$, the signs being reversed for $n+m<2$.

6. CONTINUOUS DEPENDENCE ON THE VELOCITY

We discuss in this section the dependence of the above-constructed waves on the velocity. We denote by

$$u_c(x,t)=\phi(\xi,c), \qquad \xi=ct-x \tag{6.1}$$

the FTW with velocity c. Here ξ varies in $[0,\infty)$ and c in the domain of existence which depends on n and m. Again we use the notations $X=\phi^m$ and $Y=\partial\phi^m/\partial\xi$. We have

THEOREM 6. *Both X and Y are continuous functions in (ξ, c) and strictly increasing in both variables.*

Proof. The monotonicity of Y with respect to X, of X with respect to ξ, and of Y with respect to ξ follows from formulas (3.5), (3.6). Observing that F in (3.6) is an increasing function of c (for $X>0$) we conclude that $Y=Y(X,c)$ is strictly increasing in c for every $X>0$. By (3.5), so is $X=X(\xi,c)$ and consequently $Y=Y(\xi,c)$, for every $\xi>0$.

We have already seen that X and Y are continuous functions of ξ for every c. To prove continuity in both variables, let us take a sequence c_n strictly increasing to c and such that FTW's exists for c_n and c. By monotonicity, the sequences $X(\cdot, c_n)$ and $Y(\cdot, c_n)$ are strictly increasing and bounded above by $X(\cdot, c)$, $Y(\cdot, c)$, respectively. Passing to a subsequence if necessary we obtain

$$\bar{X}(\xi) = \lim_{n \to \infty} X(\xi, c_n), \qquad \bar{Y}(\xi) = \lim_{n \to \infty} Y(\xi, c_n).$$

Since $(X(\cdot, c_n), Y(\cdot, c_n))$ solves system (3.5), it is easily concluded that $(\bar{X}, \bar{Y})$ also solves the system. Moreover $\bar{X}(0) = \bar{Y}(0)$. By our uniqueness results, then $\bar{X} = X(\cdot, c)$ and $\bar{Y} = Y(\cdot, c)$. Finally, all the functions involved are continuous and the convergence is monotone. Therefore, by Dini's Theorem, the convergence is uniform. This means that X and Y are continuous jointly in (ξ, c). ∎

Remarks. (1) Using the above arguments one can show that

$$\lim_{c \uparrow \infty} \phi(\xi, c) = +\infty, \qquad \lim_{c \downarrow -\infty} \phi(\xi, c) = 0.$$

(2) A more detailed analysis of system (3.5) shows that X and Y are in fact continuously differentiable with respect to c. We omit the calculations, which are cumbersome.

(3) In case $n + m = 2$, the waves are given by the explicit formulas (2.6), (2.7) which imply

$$X = \mu(c)\, \xi^{m/(m-1)}, \qquad Y = \frac{m\mu}{m-1}\, \xi^{1/(m-1)},$$

where μ is a C^∞ increasing function of $c \in \mathbb{R}$. Note that X and Y are not C^∞ differentiable in ξ for $\xi = 0$.

7. Summary: The Equation for the Velocity

We have discussed the existence, uniqueness, and behavior of finite travelling waves (FTW) in the model equation of diffusion with absorption (1.1) for all values of the real parametes m and n. We have shown that near the front and near infinity there are three basic profiles, those arising in the special situations of pure diffusion, pure absorption, and stationary diffusion–absorption equilibrium. In all three cases, u behaves like a power of $ct - x$ for $ct - x > 0$ and vanishes for $ct - x \leqslant 0$, c being the velocity (positive or negative) of the front.

A main conclusion of the analysis of (1.1) is that the whole picture depends first of all on the sign of $p = m + n - 2$. Indeed, if $p = 0$, the three basic profiles coincide and there is a unique formula that relates the value of $(u^{m-1})_x$ at the front of a FTW with its velocity (cf. 2.7)). The wave is stationary if $(u^{m-1})_x$ has a critical value, a heating wave (HW) if it exceeds this value, and a cooling wave (CW) otherwise.

When $p>0$ our analysis shows that HW's behave near the front as the purely diffusive waves, and CW's as the purely absorptive ones. The ranges in which they exist add to this point: we have HW's if $m>1$ (with the proviso that the wave exists globally in Q if $n \leqslant m$, and in some subdomain if $n>m$) whereas CW's exist if $n<1$ (this automatically implies $m>1$ in the current case).

On the contrary, for $p<0$ profiles near the front are in first approximation given by the equilibrium profile in all cases and, accordingly, FTW's exist whenever $|n|<m$. Now the second-order approximation contains the information about the velocity. In this way, we confirm the formal analysis of [RK] concerning the local behavior near the fronts and extend it to the whole range of existence. We remark that no FTW exists if $m \leqslant 0$, if $n \leqslant -m$, or if $m \leqslant n$ and $m \leqslant 1$.

The velocity of the wave is given by a different expression in each of the three profiles discussed. Nevertheless, it is possible to write a general formula valid in all cases. In fact, if we examine c in the ODE satisfied by the profile ϕ, we find, after using L'Hôpital's rule,

$$c = a \lim \frac{(\phi^m)'}{\phi} - b \lim \frac{\phi^n}{\phi'},$$

where the limit is taken as $\xi \to 0$, $\xi > 0$, and $f' = df/d\xi$. This can also be written as

$$c = a \lim \left(\frac{m}{m-1} \phi^{m-1} \right)' - b \lim \frac{1-n}{(\phi^{1-n})'}, \tag{7.1}$$

where we recognize at once the limits which appear for D- and A-profiles. Moreover, when $p>0$, if we have a D-profile, then $\phi^{m-1}(\xi) \cong K\xi$ for $\xi \cong 0$ and some $K>0$, and the second limit above vanishes. Conversely, for an A-profile $\phi^{1-n}(\xi) \cong K\xi$ for $\xi \cong 0$ and some $K>0$, so that the first limit vanishes. If $p=0$, we recover (2.7). Finally, when $p<0$ both limits are infinite, but their difference is finite. Formula (7.1) could be of interest in the study of interfaces appearing in the general Cauchy problem for (1.1).

Finally, we remark that formula (7.1) also holds as $\xi \to \infty$. In fact, for large values of ξ our analysis shows that the situation is completely reversed, S-profiles arising for $p>0$ and D- and A-profiles for $p<0$.

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